**Elastic electron-proton scattering**

- Beams of leptons are good tools for investigating the properties of hadrons since leptons have no substructure.

- Elastic lepton-hadron scattering can be used to measure the size of the hadron.

- Elastic scattering means that the same type of particles goes into and comes out of the scattering process.

![Diagram of Elastic Lepton-Proton Scattering](image)

Figure 68: The dominant one-photon exchange mechanism in elastic lepton-proton scattering.
The angular distribution of the particles emerging from a scattering reaction is given by the differential cross-section

$$\frac{d\sigma(\theta, \phi)}{d\Omega}$$

where

$$d\Omega = sin\theta d\theta d\phi$$

Figure 69: The definition of the solid angle $d\Omega$ in scattering experiments.

The total cross section of the reaction is obtained by integrating the differential cross section:

$$\sigma = \int \int \frac{d\sigma(\theta, \phi)}{d\Omega} d\Omega = \int_0^{\pi} \int_0^{2\pi} \frac{d\sigma(\theta, \phi)}{d\Omega} sin\theta d\theta d\phi$$
The scattering of an electron on a static point-like electrical charge.

\[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4p^4 \sin^4 \left( \frac{\theta}{2} \right)} \left( m^2 + p^2 \cos^2 \frac{\theta}{2} \right) \]

In the low energy limit \( p \ll m \), the Mott scattering formula is reduced to the non-relativistic Rutherford scattering formula:

\[ \frac{d\sigma}{d\Omega} = \frac{m^2 \alpha^2}{4p^4 \sin^4 \left( \frac{\theta}{2} \right)} \]

where \( \alpha = \frac{e^2}{4\pi} \).
If the electric charge is not point-like, but it is spread out with a spherically symmetric density distribution, i.e., $e \rightarrow e \rho(r)$, where $\rho(r)$ is normalized:

$$\int \rho(r) d^3 \tilde{x} = 1$$

then the Rutherford scattering formula has to be modified by an electric form factor $G_{E}^2(q^2)$:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_R G_{E}^2(q^2)$$  \hspace{1cm} (89)
The electric form factor is the Fourier transform of the charge distribution with respect to the momentum transfer $\vec{q}$:

$$G_E(q^2) = \int \rho(r) e^{i\vec{q} \cdot \vec{x}} d^3\vec{x} \quad (90)$$

- For $q = 0$, $G_E(0) = 1$ (low momentum transfer)
- For $q^2 \to \infty$, $G_E(q^2) \to 0$ (large momentum transfer)

Measurements of the cross-section can be used to determine the form-factor and hence the charge distribution.

The mean quadratic charge radius is for example given by

$$r_E^2 = \int r^2 \rho(r) d^3\vec{x} = -6 \left. \frac{dG_E(q^2)}{dq^2} \right|_{q^2 = 0} \quad (91)$$
Scattering of electrons on protons depend not only on the electric form factor \( G_E \) but also on a magnetic form factor \( G_M \) which is associated with the magnetic moment distribution.

\[
\frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_M \times \left( G_1(Q^2) \cos^2 \frac{\theta}{2} + \frac{Q^2}{2M^2} G_2(Q^2) \sin^2 \frac{\theta}{2} \right)
\]

\[
G_1(Q^2) = \frac{G_E^2}{1 + \frac{Q^2}{4M^2}} \quad \text{and} \quad G_2(Q^2) = \frac{G_M^2}{1 + \frac{Q^2}{4M^2}}
\]
Measurement of the form factors are conveniently divided into three $Q^2$ regions:

1) low $Q^2 \Rightarrow Q \ll M \Rightarrow G_E$ dominates the cross-section and $r_E$ can be precisely measured:

$$r_E = 0.85 \pm 0.02 \text{ fm}$$ (92)

2) An intermediate range: $0.02 \leq Q^2 \leq 3 \text{ GeV}^2 \Rightarrow$ both $G_E$ and $G_M$ give sizeable contribution $\Rightarrow$ the result can be given by the parameterization:

$$G_E(Q^2) \approx \frac{G_M(Q^2)}{\mu_p} \approx \left(\frac{\beta^2}{\beta^2 + Q^2}\right)^2$$ (93)

with $\beta^2 = 0.84 \text{ GeV}$

3) high $Q^2 > 3 \text{ GeV}^2 \Rightarrow G_M$ dominates the cross section:
The form factors are normalized so that

\begin{align*}
G_E(0) &= \text{total charge} = 1 \quad (p) \\
        &= 0 \quad (n) \\
G_M(0) &= \text{magnetic moment} = \mu_p = +2.79 \quad (p) \\
        &= \mu_n = -1.91 \quad (n)
\end{align*}

\[ \rightarrow \quad \text{If the proton is a point particle then} \]

\[ G_E(Q^2) = 1 \quad \text{and} \quad G_M(Q^2) = 1 \]
**Inelastic lepton-proton scattering**

In inelastic electron-proton scattering a new dimensionless variable called the **Bjorken scaling variable** \( x \) is introduced where \( 0 < x < 1 \).

![Diagram of one-photon exchange in inelastic lepton-proton scattering](image)

Figure 74: One-photon exchange in inelastic lepton-proton scattering.

In inelastic lepton-proton scattering a new dimensionless variable called the **Bjorken scaling variable** \( x \) is introduced where \( 0 < x < 1 \).
The differential cross section for inelastic electron-proton scattering can be written as:

$$\frac{d\sigma}{dE_2 d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\left(\frac{\theta}{2}\right)} \cdot \frac{1}{4} \cdot$$

$$\cdot \left[ F_2(x, Q^2) \cos^2\left(\frac{\theta}{2}\right) + \frac{Q^2}{xM^2} F_1(x, Q^2) \sin^2\left(\frac{\theta}{2}\right) \right]$$

The two dimensionless structure functions $F_1(x, Q^2)$ and $F_2(x, Q^2)$ parameterize the photon-proton interaction in the same way as $G_1(Q^2)$ and $G_2(Q^2)$ in elastic scattering.

Bjorken scaling or scale invariance:

$$F_{1,2}(x, Q^2) \approx F_{1,2}(x)$$

i.e. for $Q \gg M$, structure functions are almost independent of $Q^2$. If all particle masses, energies and momenta are multiplied by a scale factor, structure functions at any given $x$ remain unchanged.
Figure 75: The measured structure function $F_2$ (compilation of data from different experiments).
The first observation of scale invariance in inelastic scattering was observed at SLAC in 1969 and was later interpreted as the first evidence for quarks.

Figure 76: Two spectrometers in SLAC’s End Station A that were used to discover quarks in the late 1960s.
**Deep inelastic electron-proton scattering.**

In the parton model the scale invariance is explained by scattering on point-like constituents (partons) in the proton.

The parton model is valid if the target proton has a sufficiently large momentum, so that the fraction of the proton momentum carried by the struck quark is given by Bjorken $x$. 

\[ \mathbf{q} = \mathbf{P}_1 - \mathbf{P}_2 \]

\[ W^2 = (\mathbf{P}_3 + \mathbf{q})^2 \]
In the parton model, the structure function $F_1$ depends on the spin of the partons:

$$F_1(x, Q^2) = 0 \quad (\text{spin-0})$$

$$2xF_1(x, Q^2) = F_2(x, Q^2) \quad (\text{spin-1/2})$$

The data favours the second relation (called the Callan-Gross relation) i.e. quarks have spin 1/2.
Summary

• Quantum Chromodynamics

   a) The gauge bosons in QCD are called gluons and are spin 1 particles.

   b) The charge in QCD is called colour and gluons carry colour charge but not electric charge.

   c) The strong interaction is flavour independent.

   d) Colour confinement means that a particle with a colour charge (such as a gluon or a quark) cannot exist as a free particle.

• The strong coupling constant.

   e) The strong coupling constant $\alpha_s$ gives the strength of the strong interaction.

   f) $\alpha_s$ is not a true constant since it depends on $Q^2$. 
• **Electron-positron interactions.**

  g) Quarks are seen as jets of hadrons in electron-positron interactions.

  h) The measured cross section ratio $R$ can only be explained if there are 3 colours.

  i) A measurement of the angular distribution of jets in two-jet events show that the quark is a spin $1/2$ particle.

  j) Three-jet events can be used to measure $\alpha_s$ and to show that the gluon is a spin 1 particle.

• **Elastic electron-proton scattering.**

  k) Elastic electron-proton scattering can be used to measure the size of the proton.

  l) Scattering of electrons on protons depends on an electric and a magnetic form factor.

  m) The measurement of these form factors show that the proton is not a point particle.
• **Inelastic lepton-proton scattering.**

  n) Inelastic scattering of electrons on protons depends on two structure functions $F_1$ and $F_2$.

  o) Scale invariance means that these structure functions are almost independent on $Q^2$. The scale invariance of $F_2$ is evidence for the existence of quarks in the proton.

• **Deep inelastic electron-proton scattering.**

  p) The measurement of $F_1$ show that the quarks have to be spin 1/2 particles.